Math 112 Béla Bajnok

Final Exam Spring 2021

NAME: Andry Arthur Rakotonjanabelo

Write and sign the full Honor Pledge here:

I affirm that I have upheld the highest principles of honesty and integrity in my academic work and have not witnessed a violation of the Honor Code.

General instructions – Please read!

- The purpose of this exam is to give you an opportunity to explore a series of complex and challenging questions, gain a comprehensive view of calculus and its applications, and develop some creative writing, problem solving, and research skills.
- **All your assertions must be completely and fully justified.** At the same time, you should aim to be as concise as possible; avoid overly lengthy arguments and unnecessary components. Your grade will be based on both mathematical accuracy and clarity of presentation.
- Your finished exam should **stand alone as an article**, consisting of complete sentences, thorough explanations, and exhibit correct grammar and punctuation.
- I encourage you to prepare your exam using LaTeX. However, you may use instead other typesetting programs that you like, and you may use hand-writing or hand-drawing for some parts of your exam. In any case, **the final version that you submit must be in PDF format.**
- It is acceptable (and even encouraged) to discuss the exams with other students in your class or with the PLA. However, **you must individually write up all parts of your exams.** To elaborate:
  - You are absolutely allowed to discuss every aspect of the exams, starting with the mathematics involved and up to the details of the technical issues of typesetting.
  - You are allowed to meet with another student and show parts of your work during the discussion.
  - You are NOT allowed to send parts of your work to another student.
  - You are NOT allowed to have other students copy your work.

  - You are NOT allowed to copy another student's work.

- **You may use the text, your notes, and your homework, but no other sources.**

- You must write out a complete, honest, and detailed acknowledgment of all assistance you received and all resources you used (including other people) on all written work submitted for a grade.

- Submit your exam to me by email at bbajnok@gettysburg.edu by the deadline announced in class.

Good luck!

1 Introduction

Calculus is about the mathematical truths of how things change. It's about reducing graphs, functions, and formulas to infinitesimally single differences/intervals to know the rate of change of the graph, also called differentiation, and/or to find the total area under graphs, also called integration. These two mathematical processes are inverse concepts, which means they can cancel each other. Together, they form incredible tools that apply to most of the natural world.

The main concepts and topics of Calculus 2 are using Riemann sums to approximate and ultimately find the area under a curve, limits, infinite sequences and infinite series. Through calculus 2, we also explored how the integration concepts can go beyond 2D planes. With the help of integration rules, we can determine how these infinite sequences and series behave as we approach infinity. Special kind of infinite series, called Taylor series, are used to find approximations of the value near a given point. Through the following 5 subsections, we are going to explore 5 themes of Calculus.

2 Five Themes of Calculus

2.1 Mathematical Survivor

The Mathematical Survivor game involves a group of players staying on an island to win a one million dollar prize. Each player is numbered in order from 1 to n number of players. Each day, the player with the highest number is being voted on whether they should stay or leave. If half or more of the people vote for the designated person to leave, that player leaves and the game goes on to the next day. Otherwise, everyone left stays to get an equal share of the prize and the game ends. Let n be the number of plays on the island, d_n be the number of days a game scenario goes, and p_n the number of players at the end of a game scenario.

2.1.1 Some small values of n

The following tables in this section represent the best voting outcomes for each case of player number n from $n = 1$ to $n = 8$. Each column PN represents the vote of player P with number N. Each row (apart from the first) represents a day with the subsequent votes of each play. One vote is shown in each cell with either S for "Stay" or L for "Leave". The last column of each table shows the voting outcome for the specific day. Each player will vote either S or L according to their best interests.

Day(s)	P1	outcome
1	S	1 wins

Table 1: When $n = 1$, the game ends on the first day as P1 would logically decide to stay and make himself win.

Day(s)	P1	P2	outcome
1	L	S	2 leaves
2	S		1 wins

Table 2: When $n = 2$, the game ends in two days. Since P1 has half of the votes, P1 logically decides to make P2 leave. On the second day, P1 decides to make himself stay in order to win the prize for himself.

Day(s)	P1	P2	P3	Outcome
1	L	S	S	1, 2, 3 win

Table 3: When $n = 3$, on the first day, P3 would decide to make himself stay. Since P2 knows that if P3 leaves P1 will manage to win, P2 votes for P3 to stay. Since P3 manages to stay, the game ends on day 1 and the money is shared between them three.

Day(s)	P1	P2	P3	P4	Outcome
1	L	L	L	S	4 Leaves
2	L	S	S		1, 2, 3 win

Table 4: When $n = 4$, on the first day, all the first three players know that if P4 manages to stay they will have to share the prize between them 4 instead of them 3. P1, P2, and P3 decide to vote P4 out. On the second day, as in the previous $n = 3$ case, P2 and P3 has to vote for P3 to stay in order to get win.

Day(s)	P1	P2	P3	P4	P5	Outcome
1	L	L	L	S	S	5 leaves
2	L	L	L	S		4 leaves
3	L	S	S			1, 2, 3 win

Table 5: When $n = 5$, following the same logic as when $n = 4$, P1, P2 and P3 all decide to vote P5 out in order to keep the prize among themselves. No matter what P4 and P5 votes, P5 is voted out. On the second day, P1, P2, and P3 vote P4 out, and win on the 3rd day.

Day(s)	P1	P2	P3	P4	P5	P6	outcome
1	L	L	L	S	S	S	6 Leaves
2	L	L	L	S	S		5 Leaves
3	L	L	L	S			4 Leaves
4	L	S	S				1, 2, 3 win

Table 6: When $n = 6$, similar to the last 2 games, P1, P2, and P3 still hold at least half of the votes. They end up winning after 4 days.

2.1.2 Values for $n = 1 \ldots 30$

If we follow the same process for the next n cases until $n = 30$, the pattern starts to seem clearer.

In Table 9 you can see that as n increases from 1 to 30, p_n only increases a few times.

Day(s)	P1	P2	P3	P4	P5	P6	P7	outcome
1	L	L	L	S	S	S	S	1 , . . . , 7 win

Table 7: When $n = 7$, P7 logically votes to make himself stay. P6, P5, and P4 know that if P7 leaves, they lose all their chances to win a part of the prize. They vote to make P7 stay. P1, P2 and P3 would of course want P7 to leave in order to keep most of the money among themselves. As P4, P5, P6 and P7 have most of the votes, P7 manages to stay and the game ends.

Day(s)	P1	P2	P3	P4	P5	P6	P7	P8	outcome
1	L	L	L	L	L	L	L	S	8 Leaves
2	L	L	L	S	S	S	S		1, . . . , 7 win

Table 8: When $n = 8$, the game ends in two days. The first 7 players decide to vote 8 out on the first day as they know they could win a greater part of the prize with P8. The outcome of the second day is then exactly as when $n = 7$, since this is the best outcome.

Notice also that d_n gradually increases until it goes back to 1 when p_n changes. p_n is equal to n only when n is large enough to have another majority decide of the outcome of the game. From $n = 1$ to $n = 2$, P1 had at least half of the voting power. When $n = 3$, P1 no longer had the half of the voting power as P2 and P3 had the majority. From $n = 3$ to $n = 6$, P1, P2 and P3 still kept at least half of the votes so they keep sharing the prize among themselves, which means from $n = 3$ to $n = 6$, $p_n = 3$.

As n increases, p_n remains equal to the number of players keeping half or more of the votes until n reaches a point where the majority changes. When $n = 7$, P1, P2, P3 can no longer decide the outcome of the game on their own as P4, P5, P6 and P7 took the majority from them. The number of remaining players, p_n will stay 7 until there are more than 7 other players to take away the voting power from the first 7 players. This is the pattern of the game for n number of players. We can now predict the next values of n where p_n changes:

$$\{p_n\} = \{3, 7, 15, 31, 63, 127, 255, 511, \dots\}.$$

We can deduce that when n is between those values, p_n maps down to the last value where the change in majority happened.

Figure 1 visually shows the behavior of the pattern of p_n as n increases.

n	Length of game (d_n)	Remaining People (p_n)
1	1	1
2	2	1
3	1	3
4	2	3
5	3	3
6	4	3
7	1	7
8	2	7
9	3	7
10	4	7
12	5	7
13	6	7
14	7	7
15	1	15
16	2	15
17	3	15
18	4	15
19	5	15
20	6	15
21	7	15
22	8	15
23	9	15
24	10	15
25	11	15
26	12	15
27	13	15
28	14	15
29	15	15
30	16	15

Table 9: $n = 1$ to $n = 30$

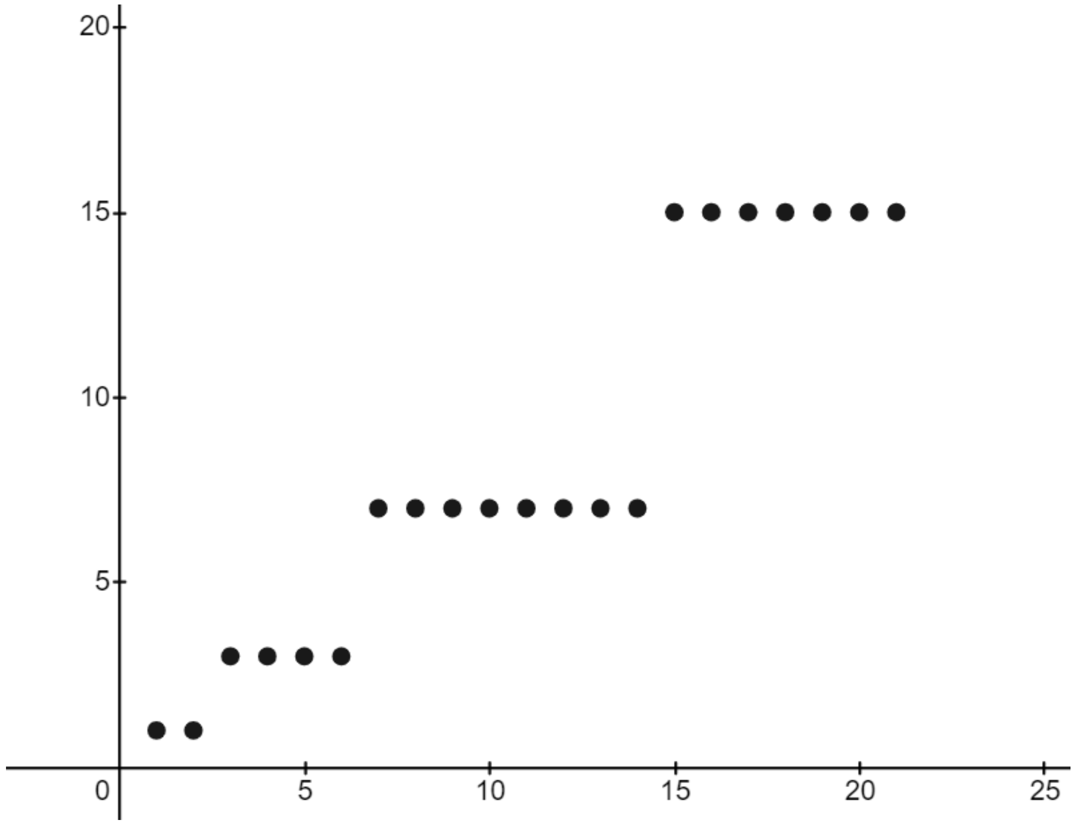


Figure 1: Graph of p_n for first 20 people.

2.1.3 A General Solution

In order to find the formula, we need to first look at the specific cases of n where p_n changes.

$$\begin{array}{rcccc} n = & 1,\ldots & 3,\ldots & 7,\ldots & 15\ldots \\ p_n = & 1, & 3, & 7, & 15\ldots \end{array}$$

If we calculate the differences between the terms above, we get 2, 4, 8, 16... There is a multiplication of 2 in the difference as the terms increase. This means it's a geometric sequence with the common ratio of 2. To get the exact values of the terms, we need to subtract those differences by 1: 4 - 1 = 3, 8 - 1 = 7, 16 - 1 = 15. . . . We can then deduce an the "overall" (but still incomplete) form of the formula to look like

$$p_n = 2^k - 1,$$

with k, as function of n, still needing to be determined.

To find what k stands for, we can use this form of the functions of the sequence p_n for the cases where it changes: $p_n = 2^{n+1} - 1$, where n, here, is the number terms of this specific sequence.

From here, we know that at some points, $n = p_n$. To get such output, the common ratio of 2 needs to be cancelled. The only operation that can cancel the base 2 of which $n + 1$ is

$$\begin{array}{rccccc} n = & 1, & 2, & 3, & 4, & 5\ldots \\ p_n = & 3, & 7, & 15, & 31, & 63\ldots \end{array}$$

the exponent is to have *log* of the same base (2) of $n+1$ in the exponent:

$$p_n = 2^{log_2(n+1)} - 1.$$

Now that the cases of $n = p_n$ are accounted for, we need to account for the cases where p_n keeps latest value of n at which p_n was able to change, referring to the behavior as Figure 1 shows.

We know another function who has a similar behavior as our game pattern: the "floor" function. The floor function outputs the greatest integer less than or equal to the input. We then need to input the exponent $log_2 n + 1$ inside the floor function. In this way, p_n will map to the greatest integer at which n has changed the majority and the best outcome. Going back to our original equation:

$$p_n = 2^k - 1,$$

where $k = \text{floor}(log_2(n+1))$. The final function of p_n for the number of players n is

$$p_n = 2^{floor(log_2(n+1))} - 1.$$

2.2 Sum(o) Wrestling

Basing on the theorem : (1) If f is integrable on $[\ a, b\ ]$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $x_i = a + i\Delta x$ and $\Delta x = \dfrac{b-a}{n}$, we can prove that

$$\int_a^b x^k\ \mathrm{d}x = \frac{1}{k+1}(b^{k+1} - a^{k+1})$$

holds for $k = 1$ and $k = 2$.

2.2.1 Integrals as limits of Riemann Sums

Assuming that the integral of a function $f(x)$ from a to b represents the area under the graph of $f(x)$ from $x = a$ to $x = b$, there should be a way to express it through some sort of area formula. The most basic area formula, which is of the rectangle, is

$$A = \text{base} \cdot \text{height}.$$

However, the graph of a function seldom resembles a 4-sided shape. A fundamental idea behind Riemann sums is to fit n numbers of rectangles under the graph between $x = a$ and $x = b$ and take the areas of each of them. If we added up the areas of these rectangles, we would get an approximate value of the area under the graph. The total area of all the rectangles is

$$\text{Total Area} = A_1 + A_2 + A_3 + A_4 + \cdots + A_n$$

To determine the height of a rectangle, we would need to have the value of $f(x)$ at the x value of the height we are trying to find out. As we are trying to fit as many rectangles between $[a, b]$, each of them has to take the same amount in base, so the base would be the total value between $x = a$ and $x = b$ divided into n equal-based rectangles, or

$$\text{base} = \frac{b-a}{n}.$$

The more rectangles we have, the smaller the bases and the closer the sum of our rectangles' areas gets to the actual value of the area. Since we are going to sum up the areas of each rectangle, we can format what we know so far into the summation notation

$$\text{Total Area} = \sum_{i=1}^{n} (\text{height}) \cdot \frac{b-a}{n}.$$

The height can be expressed by the value of $f(x)$ at x_i for the x at the i-th rectangle's base/sub interval, or

$$\text{height} = f(x_i)$$

More precisely, x_i starts at a and up with every subsequent bases up to the base of the i-th rectangle, or

$$x_i = a + \frac{b-a}{n} i$$

Since we have the representation of the total Area to

$$\text{Total Area} = \sum_{i=1}^{n} f(x_i) \cdot \frac{b-a}{n}.$$

In order to get the actual area under the graph, we would need to have infinitely many rectangles with infinitesimal bases to get the approximation as close as possible. We could express this in the limit:

$$\int_a^b x \, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \frac{b-a}{n}$$

or, with the base expressed in terms of x to be Δx,

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

This means that the integral from $x = a$ to $x = b$ of $f(x)$ is equal to the limit as n approaches infinity of the sum, starting with $i = 1$ until n, of $f(x_i)$ for the height of the function at x_i multiplied by the base or Δx.

2.2.2 For $k = 1$

Applying (1) we get

$$\int_a^b x \, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{b-a}{n} i\right) \frac{b-a}{n},$$

where $f(x) = x^1 = x$, $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + \dfrac{b-a}{n} i$. We now only need to rearrange and simplify using algebra:

$$\begin{aligned}
\int_a^b x \, \mathrm{d}x &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left(a + \frac{b-a}{n} i\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(\sum_{i=1}^{n} a + \sum_{i=1}^{n} \frac{b-a}{n} i\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a \sum_{i=1}^{n} 1 + \frac{b-a}{n} \sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a(n) + \frac{b-a}{n} \left(\frac{n^2}{2} + \frac{n}{2}\right)\right) \\
&= \lim_{n\to\infty} ab - a^2 + \frac{(b-a)^2}{2} + \frac{(b-a)^2}{2n} \\
&= ab - a^2 + \frac{(b-a)^2}{2} + 0 \\
&= ab - a^2 + \frac{b^2 - 2ab + a^2}{2} \\
&= ab - a^2 + \frac{b^2}{2} - ab + \frac{a^2}{2} \\
&= \frac{b^2}{2} + \frac{a^2}{2} - a^2 = \frac{b^2}{2} + \frac{a^2}{2} - \frac{2a^2}{2} \\
&= \frac{b^2}{2} - \frac{a^2}{2} = \frac{1}{2}(b^2 - a^2)
\end{aligned}$$

Thus it is safe to say that

$$\int_a^b x^1 \, \mathrm{d}x = \frac{1}{2}(b^2 - a^2) = \frac{1}{1+1}(b^{1+1} - a^{1+1})$$

for $k = 1$.

2.2.3 For $k = 2$

Applying (1) we get

$$\int_a^b x \, \mathrm{d}x = \lim_{n\to\infty} \sum_{n=1}^{\infty} \left(a + \frac{b-a}{n} i\right)^2 \frac{b-a}{n},$$

where $f(x) = x^2$, $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + \dfrac{b-a}{n} i$. Following the same procedure as in the previous section, we get:

$$\begin{aligned}
\int_a^b x^2 \, \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{b-a}{n} i\right)^2 \frac{b-a}{n} \\
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left(a + \frac{b-a}{n} i\right)^2 \\
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left(a^2 + \frac{2a(b-a)}{n} i + \frac{(b-a)^2}{n^2} i^2\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(\sum_{i=1}^{n} a^2 + \sum_{i=1}^{n} \frac{2a(b-a)}{n} i + \sum_{i=1}^{n} \left(\frac{(b-a)^2}{n^2} i^2\right)\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a^2 \sum_{i=1}^{n} 1 + \frac{2a(b-a)}{n} \sum_{i=1}^{n} i + \frac{(b-a)^2}{n^2} \sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a^2(n) + \frac{2a(b-a)}{n} \left(\frac{n^2}{2} + \frac{n}{2}\right) + \frac{(b-a)^2}{n^2} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right) \\
&= \lim_{n\to\infty} a^2 b - a^3 + a(b-a)^2 + \frac{a(a-b)^2}{n} + \frac{(b-a)^3}{3} + \frac{(b-a)^3}{2n} + \frac{(b-a)^3}{6n^2} \\
&= a^2 b - a^3 + a(b-a)^2 + 0 + \frac{(b-a)^3}{3} + 0 + 0 \\
&= a^2 b - a^3 + a(b-a)^2 + \frac{(b-a)^3}{3} \\
&= a^2 b - a^3 + ab^2 - 2ba^2 + a^3 + \frac{b^3 - 3b^2 a + 3ba^2 - a^3}{3} \\
&= ba^2 - a^3 + ab^2 - 2ba^2 + a^3 + \frac{b^3}{3} - ab^2 + ba^2 - \frac{a^3}{3} \\
&= \frac{b^3}{3} - \frac{a^3}{3} = \frac{1}{3}(b^3 - a^3)
\end{aligned}$$

Therefore,

$$\int_a^b x^2 \, \mathrm{d}x = \frac{1}{3}(b^3 - a^3) = \frac{1}{2+1}(b^{2+1} - a^{2+1})$$

for $k = 2$.

2.3 Different Viewpoints

If we were to figure the shape of a three-dimensional object with its front view is a square, its side view is a triangle, and its top view is a circle, we would notice that two different solids can both fulfill these conditions. Despite having showing the same shapes from the same corresponding viewpoints, these two solids have different cross sections (with respect to the same direction) and hence have two different volumes.

2.3.1 First object

Cross sections of the object in a given direction

An illustration of the major geometric components of the first object is shown below:

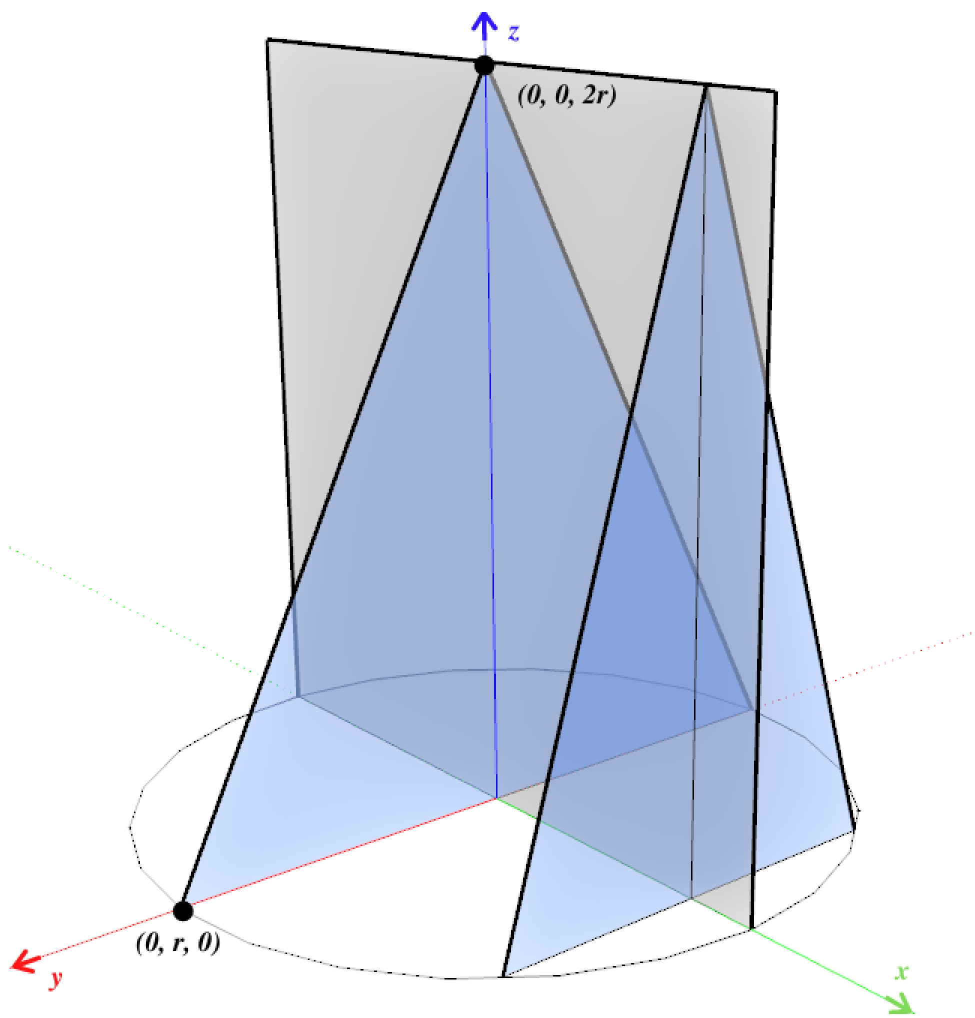


Figure 2: Illustration of first object with its major features

The base of this object is a circle. Looking towards the object along the direction of the x-axis, we see the triangular aspect of the object. Examining the cross sections in this direction, we notice that they each form triangles. These triangles of have the same height but they have different bases. The bases of each triangle form chords on the circle. The length of each chord, which is it's base, is dependent on the y value of circle at the given x (This will be demonstrated further in the process of finding the object's volume). In simple terms, the closer the cross section is to the origin, the greater its base. This means that the cross section at the origin is a triangle with the same height as the others but the largest base as its base is the the circle's diameter.

For this object to fulfill all the set conditions, however, the height of each triangle needs to imperatively be twice the length of the radius. This allows for the front/side view of the object to be a square with all sides equal to the diameter of the circle.

Volume of the object

Using the information about the cross section from the previous section we can find the volume of this solid using the formula:

$$\mathrm{V} = \int_a^b A(x)\mathrm{d}x.$$

This formula states that the volume of an object can be expressed by the integral of the areas of the cross sections of the object with respect to the x-axis from $x = a$ to $x = b$. Since the cross sections are in the shapes of triangles, we can use the formula of the area of the triangle as a basis for $A(x)$:

$$\mathrm{A}(x) = \frac{1}{2} \cdot \mathrm{base} \cdot \mathrm{height}.$$

As determined earlier, the height of all the cross sections is equal to twice the length of the radius. However, the base of a given cross section at x is twice the value of the y of this specific x. From the formula $x^2 + y^2 = r^2$, we get $y = \sqrt{r^2 - x^2}$. Multiplying the result by two will give us the base at the given x. We can then rewrite the Area as

$$\mathrm{A}(x) = \frac{1}{2} \cdot 2\sqrt{r^2 - x^2} \cdot 2r = 2r\sqrt{r^2 - x^2}.$$

The formula for the volume is then

$$\mathrm{V} = \int_{-r}^{r} 2r\sqrt{r^2 - x^2}\mathrm{d}x,$$

with the limits of integration being from $x = -r$ to $x = r$ as we are integrating in terms of the radius of the top view circle. Evaluating the integral we get

$$
\begin{aligned}
V &= \int_{-r}^{r} 2r\sqrt{r^2 - x^2}\mathrm{d}x \\
&= 2\int_{0}^{r} 2r\sqrt{r^2 - x^2}\mathrm{d}x \\
&= 4r\int_{0}^{r} \sqrt{r^2 - x^2}\mathrm{d}x
\end{aligned}
$$

The given formula results in the shape of the semicircle as shown below. Deducing that the shaded region makes one-fourth of the area of the circle, the volume can be found using the formula of the area of the circle, $\mathrm{A}(x) = \pi r^2$:

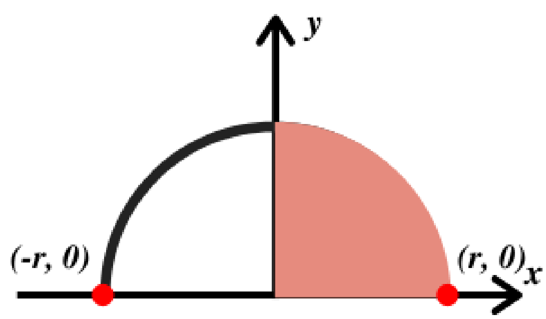


Figure 3: Semicircle resulting from the formula, with the shaded region showing what is to be integrated

$$
\begin{aligned}
&V = 4r(\frac{1}{4} \cdot \pi r^2) \\
&V = \pi r^3
\end{aligned}
$$

2.3.2 Second object

Cross sections of the object in a given direction

Even though this second object has the same corresponding views as the first one, the potent differences account for the two objects having two different volumes. Below is the equivalent illustration of the important geometric components of the second object:

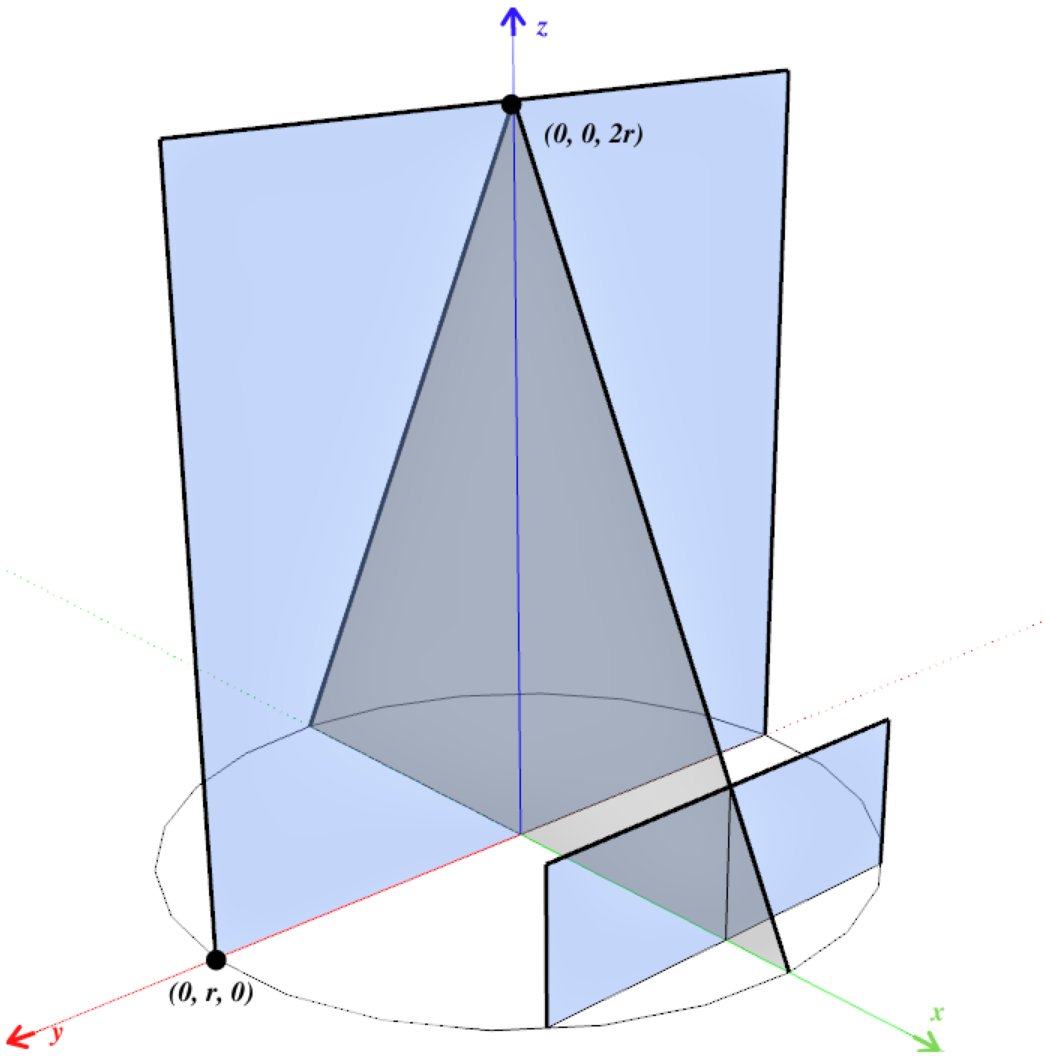


Figure 4: Illustration of second object with its major features

Looking at the cross sections from the same direction as the first one (the direction corresponding to the x-axis), we notice rectangles of different shapes and sizes as cross sections. The closer the cross section is the middle of the circle, at $x = 0$, the greater its length and width. The cross section at $x = 0$ forms the predominant square with the side lengths of $2r$ which allows for the square view from the front. Perpendicular to this square forms the image of a rectangle with the base equal to its height, which is equal to $2r$ as well. The widths (or bases) of the rectangular cross sections are dependent on the value of y at the given x on the circle while their height are set by the value of z at the given x of the triangle.

Volume of the object

We know that the volume is represented by

$$\mathrm{V} = \int_a^b A(x)\mathrm{d}x.$$

In this case, the area of the cross sections should resemble that of the rectangle as opposed to the one of the triangle in the first object:

$$\mathrm{A}(x) = \text{width} \cdot \text{length}.$$

The width of the given cross section, similar to the base of the cross sections of the previous one, is equal to twice value of y of the circle at a given x (base $= 2\sqrt{r^2 - x^2}$). On the other hand, the length of this given cross section is the value of z on the triangle at the given x. We can get the formula for z with respect to x:

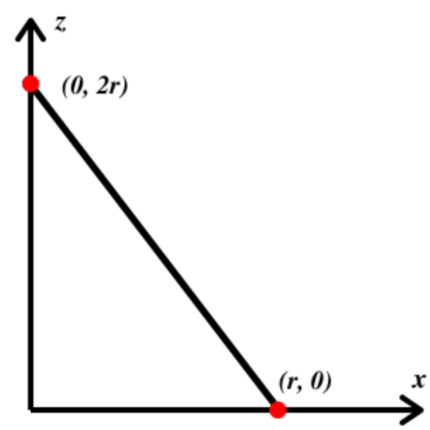


Figure 5: The line show the relationship between x and z.

From the information from figure 5, we get $z = \frac{0-2r}{r-0}x + 2r$, or

$$z = -2x + 2r.$$

Therefore, the complete formula for the rectangle cross section at a given x is

$$\mathrm{A}(x) = (2r\sqrt{r^2 - x^2})(-2x + 2r).$$

So we now have the integral to evaluate:

$$
\begin{aligned}
\mathrm{V} &= 2\int_0^r (2\sqrt{r^2-x^2})(-2x+2r)\mathrm{d}x \\
&= 2\int_0^r \left(-4x\sqrt{r^2-x^2}+4r\sqrt{r^2-x^2}\right) \\
&= 2\left(\int_0^r -4x\sqrt{r^2-x^2}\mathrm{d}x+\int_0^r 4r\sqrt{r^2-x^2}\mathrm{d}x\right) \\
&= 2\left(-4\int_0^r x\sqrt{r^2-x^2}\mathrm{d}x+4r\int_0^r \sqrt{r^2-x^2}\mathrm{d}x\right) \\
&= 2\left(-4\left[-\frac{1}{3}(r^2-x^2)^{\frac{3}{2}}\right]_0^r+4r\left(\frac{\pi r^2}{4}\right)\right) \\
&= 2\left(-4(0-\left(-\frac{r^3}{3}\right))+\pi r^3\right) \\
&= 2\left(-\frac{4r^3}{3}+\pi r^3\right) \\
&= 2\pi r^3-\frac{8r^3}{3}
\end{aligned}
$$

2.4 Solitaire Army

The game Solitaire Army is a particular version of Peg Solitaire; it was discovered by the British mathematician John Conway in 1961. Peg Solitaire is played on a board that has holes arranged in a rectangular grid-like fashion (like squares on an infinite chess-board). Each hole can hold one peg. A move consists of a jump by one peg over another peg which is next to it horizontally or vertically (but not diagonally); the peg jumped over will then be removed from the board. Each move therefore reduces the total number of pegs by one.

In the game Solitaire Army, the board is an infinite plane where one horizontal line is distinguished; it's called the demarcation line. At the start of the game, all pegs are on one side of the demarcation line.

This game has the special characteristic described as such: a peg can reach 1, 2, 3, and 4 holes into the other side of the demarcation line, but cannot go any holes beyond. We are going to demonstrate, through the use of geometric series, why no pegs can go beyond the fourth row beyond the demarcation line.

2.4.1 Going beyond demarcation line

The board of the Solitary army game will be illustrated as such:

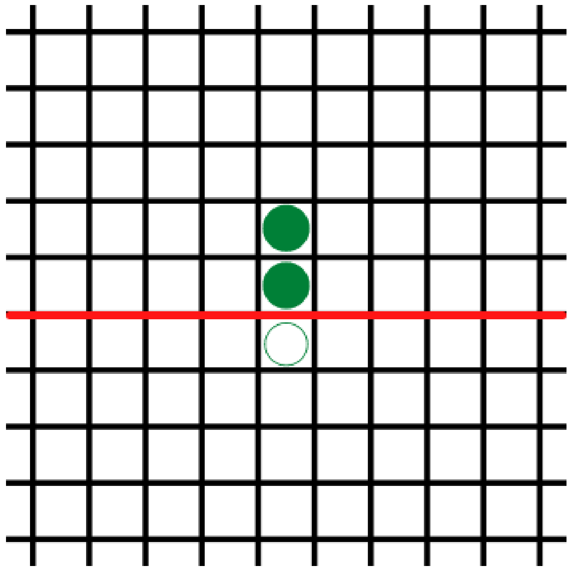


Figure 6: Illustration of the game board. Each cell represents one hole. The horizontal red line represents the demarcation line. One full green circle represent a soldier in its cell. The empty circle with green outline represents the farthest a peg can go to given the current disposition of the army.

1 row passed demarcation line

Since the board is an infinite plane, it does not have a set size. This illustration shows a portion of the board somewhere along the demarcation line.

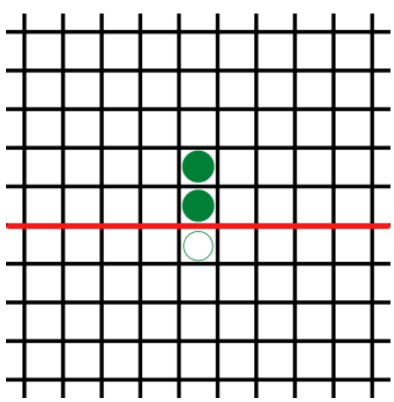


Figure 7: Set up of game board to arrive to first row.

For one peg to get across to the first row of the demarcation line, two pegs just need need to be put in the first two cells above the goal. The farthest peg from the line just needs to jump down over the other peg and into the desired hole.

2 rows passed demarcation line

Using a disposition such as the one below allows a peg to arrive to the second row passed the demarcation line:

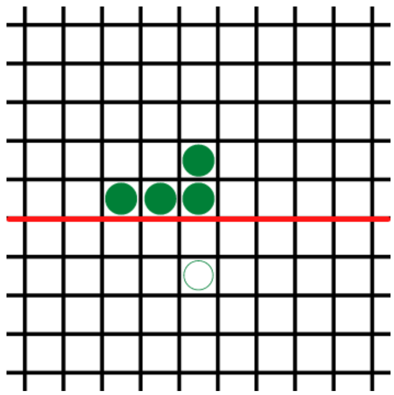


Figure 8: Set up of game board to arrive to second row.

From this disposition, we can reach the second row as such:

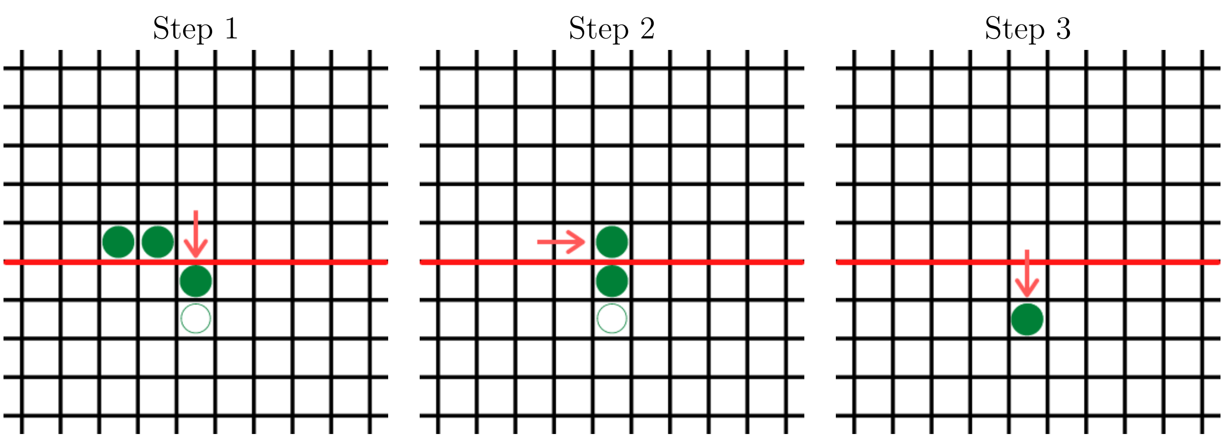


Figure 9: Steps, from left to right, to reach second row.

3 rows passed demarcation line

Using the set up below, it is possible to reach the third row:

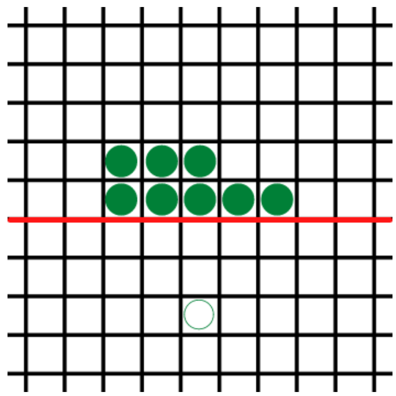
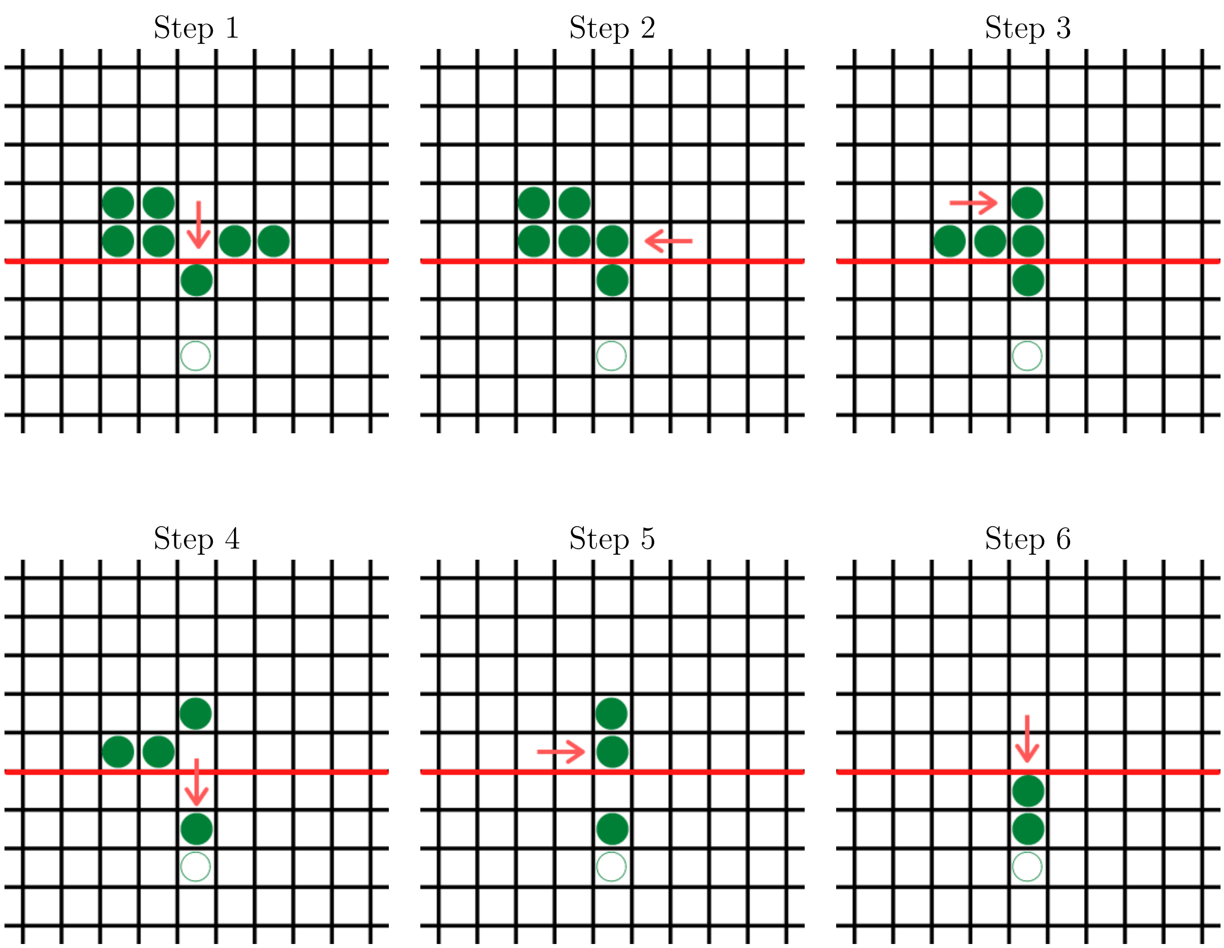
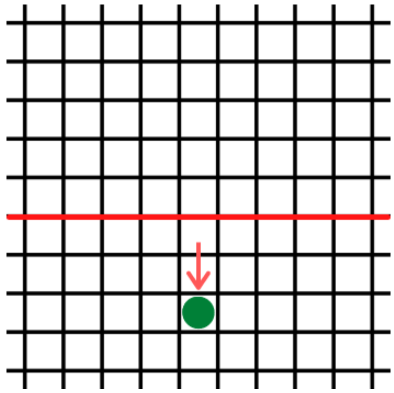


Figure 10: Set up of game board to arrive to third row.



Step 7



4 rows passed demarcation line

This is one of the disposition that can allow the peg to achieve the fourth row:

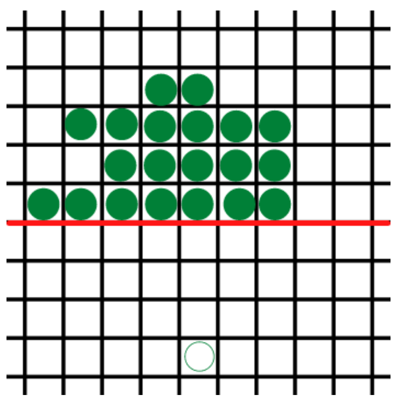
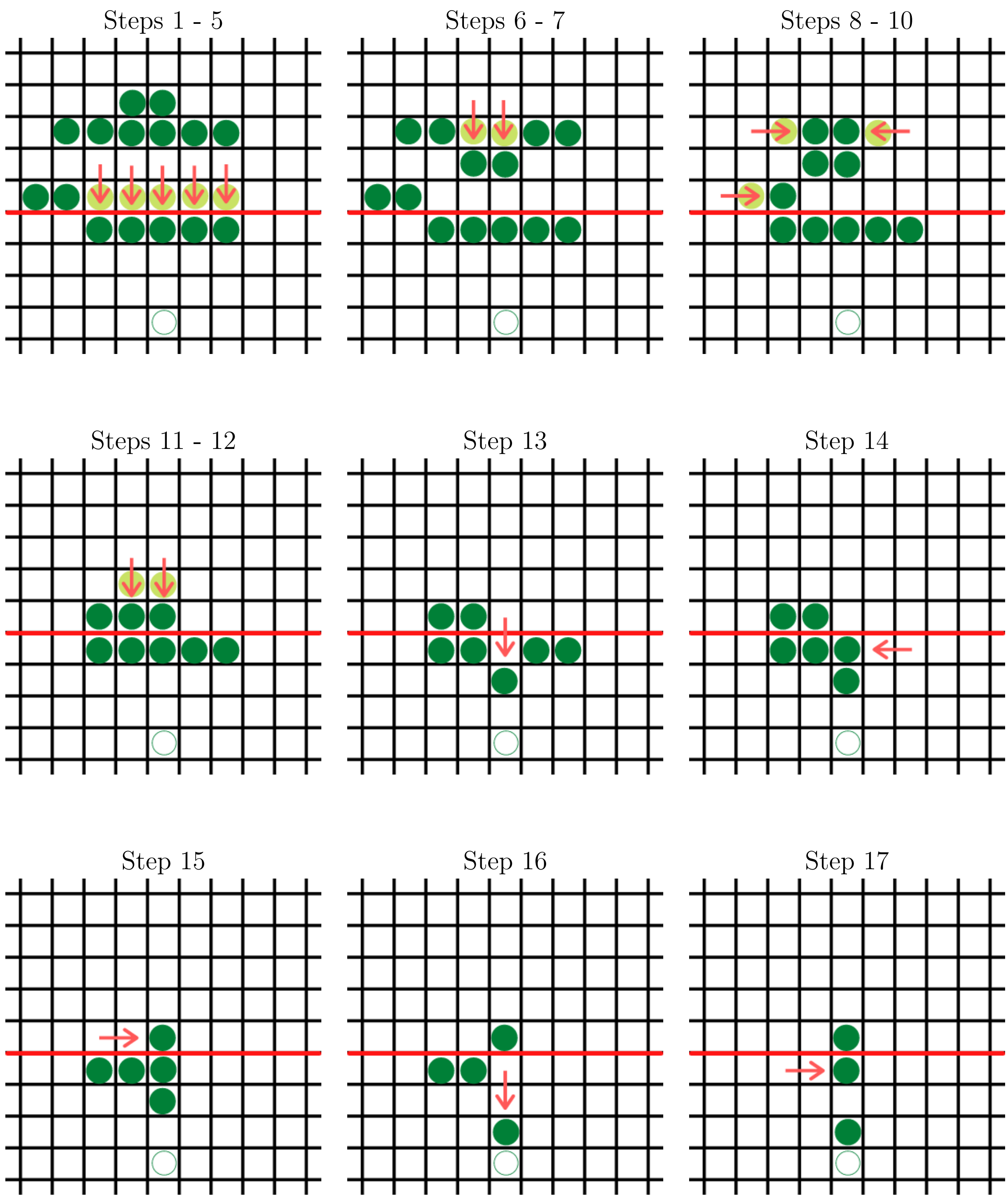
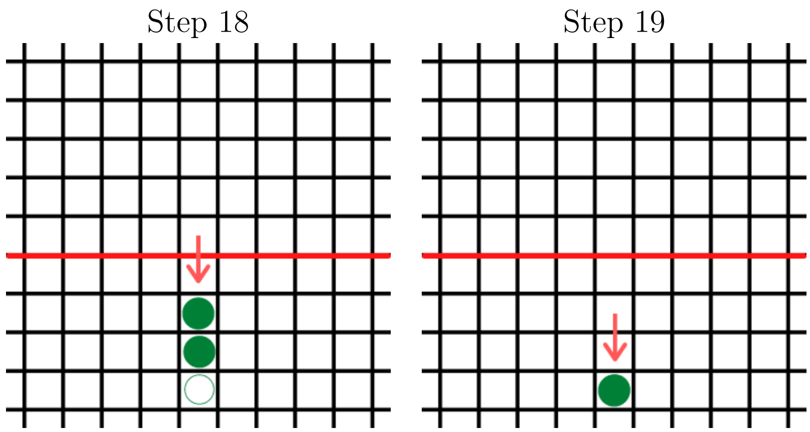


Figure 11: Set up of game board to arrive to fourth row.

As this fourth scenario comprises many more steps than the previous ones, similar steps will be combined in one drawing. In those cases, the lighter green circles show which pegs are getting jumped over for it to may be clear. It is also important to note that the steps in one group/diagram can be taken in any order as long as it reaches the disposition of the following group of steps:


Steps 1 - 5
Steps 6 - 7
Steps 8 - 10
Steps 11 - 12
Step 13
Step 14
Step 15
Step 16
Step 17



The last seven steps were all individualized as these steps have to follow a good order to be able to get to the fourth hole. Through these sequences of diagrams, it has been demonstrated that it is possible to get up to 4 holes beyond the demarcation line. Now we are going to demonstrate why it is mathematically impossible to get any step beyond this fourth row.

2.4.2 Impossibility through Geometric Series

To show that it is mathematically impossible to get to the fifth hole beyond the demarcation line, it is important to explain the mathematical basis of this infinite board. Let us attribute values to represent the worth of each peg in a given cell:

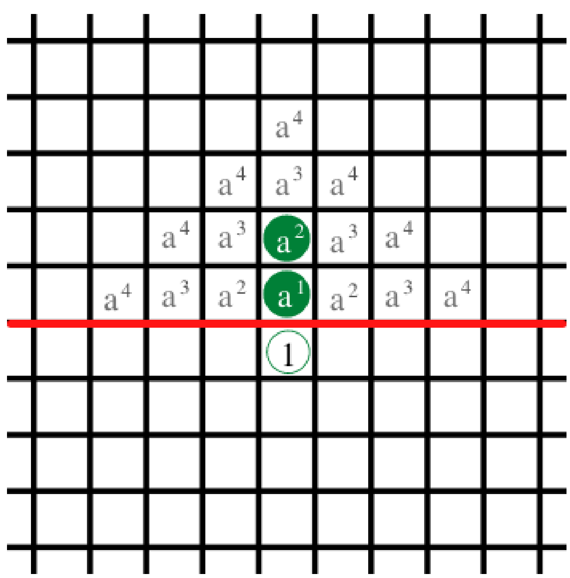


Figure 12: Values per cell.

As shown in Figure 7, let us set the value of the cell we want to attain equal to 1. As we go further from that designated cell, we multiply the value of each cell by a. In the case of the disposition in, the closest peg has value of a and the one above of a^2. As the peg above the other one is one cell farther from the goal, its value is multiplied one more by a. Let us also determine the mathematical meaning of the pegs jumping over another to get to the goal (whether it is horizontal or vertical) to be an *addition* operation. Using these definitions, move of the a^2 peg jumping over the a peg to land into the 1-valued cell can be written as

$$a + a^2 = 1.$$

Let us use the Figure 5 scenario as another example to show that this holds up:

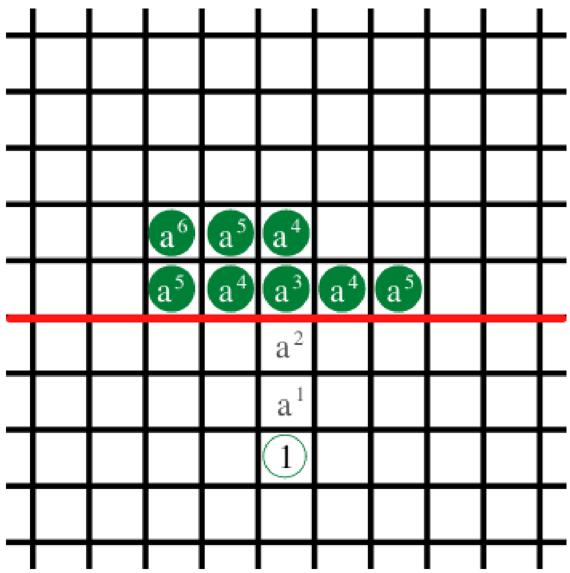


Figure 13: Values of the pegs set up to attain the third row.

All the values of the values should sum up to 1, as such:

$$\begin{aligned}
1 &= a^6 + a^5 + a^5 + a^5 + a^4 + a^4 + a^4 + a^3 \\
1 &= a^6 + 3a^5 + 3a^4 + a^3 \\
1 &= (a + a^2) \cdot (a^4 + 2a^3 + a^2) \\
1 &= (a + a^2) \cdot (a + a^2) \cdot (a + a^2) \\
1 &= (a + a^2)^3 \\
1 &= (1)^3
\end{aligned}$$

Now that we've demonstrated the application of the definitions through this example, we can apply them in an infinite soldiers scenario:

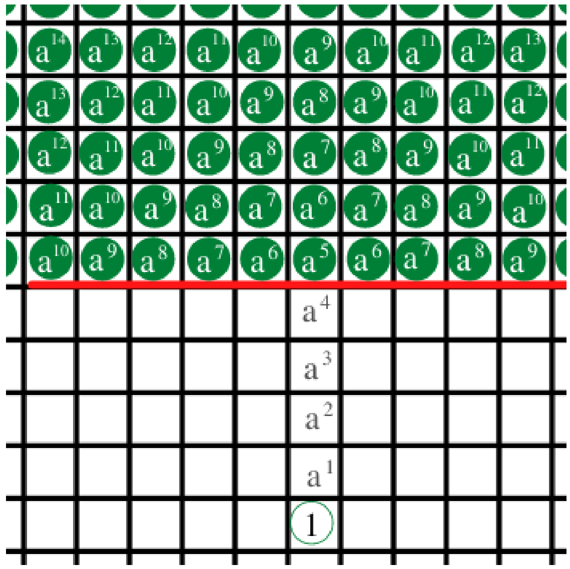


Figure 14: Game board with infinite soldiers.

In order to simplify our summation, let us separate it as such:

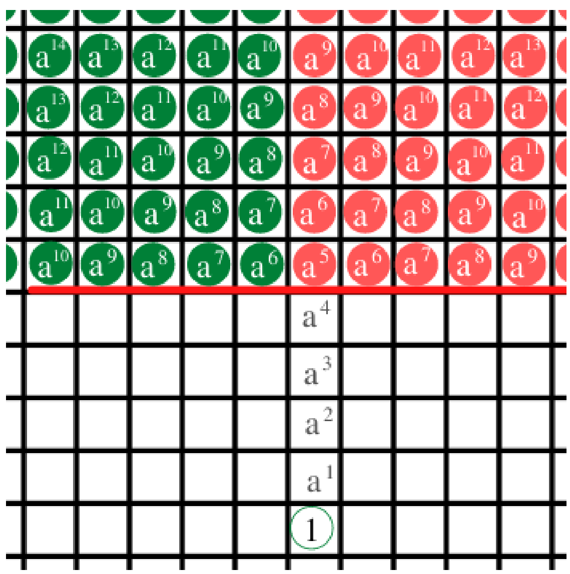


Figure 15: Adding up all the red pegs and the green pegs separately.

We will add the pegs of the row on the red side to see how much the rows are worth. We can then add the values of the rows to know the definite value of all the red soldiers, starting with the first row above the demarcation line:

$$1^{\text{st}} \text{ row} = a^5 + a^6 + a^7 + a^8 + a^9 + \dots$$

As we multiply with the common ratio of a every term, this sum occurs to be a geometric series:

$$1^{\text{st}} \text{ row} = a^5 \sum_{n=1}^{\infty} a^{n-1}.$$

In the definition of a geometric series, we know that it converges to $\frac{A}{1-r}$, where A is the first term and r is the common ratio if $-1 < r \leq 1$. In our case, the common ratio a and a has to be between -1 and 1 for $a + a^2 = 1$ to stay true. We can then calculate the sum of the first row as

$$a^5 \sum_{n=1}^{\infty} a^{n-1} = a^5 \cdot \frac{1}{1-a}.$$

Given that $a + a^2 = 1$, we can rearrange to get $1 - a = a^2$, which allows us to get

$$a^5 \sum_{n=1}^{\infty} a^{n-1} = a^5 \cdot \frac{1}{1-a} = \frac{a^5}{a^2} = a^3.$$

Following the same process, we can deduce the pattern of values as we go up in number of rows above the line:

$$\begin{aligned}
2^{\text{nd}} \text{ row} &= a^6 + a^7 + a^8 + a^9 + \cdots = \sum_{n=6}^{\infty} a^n = \frac{a^6}{a^2} \\
3^{\text{rd}} \text{ row} &= a^7 + a^8 + a^9 + a^10 + \cdots = \sum_{n=7}^{\infty} a^n = \frac{a^7}{a^2} \\
4^{\text{th}} \text{ row} &= a^8 + a^9 + a^10 + a^11 + \cdots = \sum_{n=8}^{\infty} a^n = \frac{a^8}{a^2} \dots
\end{aligned}$$

We can notice the pattern of the sum pf values of each row forming its own geometric series:

$$a^3 + a^4 + a^5 + a^6 + \cdots = \sum_{n=3}^{\infty} a^n = a.$$

The whole red portion of the army seen in Figure 10 is then valued to a. The other green portion is very similar to the red part. The only difference is that the green's first term on its first row starts with a^6, which ends up being exactly like the second row of the red portion. This means that the sum of the value of its rows starts with a^4 and follows on the pattern. The total value of the green portion of the infinite army is then

$$a^4 + a^5 + a^6 + a^7 \cdots = \sum_{n=4}^{\infty} a^n = a^2.$$

Summing up the two total values of each portion, we get $a + a^2$, which equates to 1.

The final value whole infinite army, after all the summations, ends up being 1. This means even with the infinite amount of men, we barely reach the fifth row. However, as in a true scenario we can never reach an army of infinite soldiers, it is therefore impossible get beyond the fourth row.

2.5 Some Serious Series

Let us define the sequence

$$(a_n)_{n=1}^{\infty} = \left(\frac{1}{10}, -\frac{\pi^2}{100}, +\frac{\pi^4}{1000}, -\frac{\pi^6}{10000}, + - \cdots \right)$$

and the function

$$f(x) = \frac{1}{10 + x^2}.$$

2.5.1 From infinite sequence to Maclaurin series

The sequence a_n can also be rewritten explicitly as:

$$a_n = (-1)^{n-1} \cdot \frac{(\pi^2)^{n-1}}{10^n}.$$

The "$(-1)^{n-1}$" is accounting for the alternating characteristic of the sequence. "$(\pi^2)^{n-1}$" shows how as n increases, the power of π increases by 2. "10^n" expresses how the denominator exponentially increases by the power of n at the given nth term. As this terms of the sequence alternates as n increases, the sequence seem to be slowly approaching 0. To confirm this, we need to find $\lim_{n\to\infty} a_n$.

In order to find the $\lim_{n\to\infty} a_n$ let us use the theorem:

$$\text{If} \lim_{n\to\infty} |a_n| = 0, \text{then} \lim_{n\to\infty} a_n = 0.$$

$$\lim_{n\to\infty} \left| (-1)^{n-1} \cdot \frac{(\pi^2)^{n-1}}{10^n} \right| = \lim_{n\to\infty} \frac{(\pi^2)^{n-1}}{10^n}$$

$$\lim_{n\to\infty} \frac{(\pi^2)^{n-1}}{10^n} = 0.$$

This sequence converges to 0 but it is possible to find to what value the terms infinite sequence add up to.We can notice that a_n is a geometric sequence. When we add each term of this sequence, we get a geometric series. Since it's a geometric sequence, we know that we can that it converges to

$$\frac{A}{1-r},$$

where A is the first term and r is the common ratio as long as $-1 < r \leq 1$. The sum of each term of this sequence is then:

$$\sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{(\pi^2)^{n-1}}{10^n} = \frac{1/10}{1-\frac{\pi^2}{10}} = \frac{1}{10-\pi^2}.$$

We can notice that $\dfrac{1}{1-x}$ is remotely similar in structure as $f(x) = \dfrac{1}{10+x^2}$.

We know also that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 \dots$$

Applying modifications to $\dfrac{1}{1-x}$ to get $f(x)$ can give us $f(x)$'s Maclaurin polynomial $P(x)$ as we apply the consequent modifications to original polynomial. By multiplying $\dfrac{1}{10}$ to and changing $x = -\dfrac{x^2}{10}$ from $\dfrac{1}{1-x}$ gives us $f(x)$:

$$\frac{1}{10} \cdot \frac{1}{1-\left(\dfrac{-x^2}{10}\right)} = \frac{1}{10+x^2}.$$

We now have to apply the modifications to the initial Maclaurin series to get the $P(x)$:

$$\begin{aligned}\frac{1}{10-x^2} &= \frac{1}{10} \cdot 1 + \frac{1}{10} \cdot (-\frac{x^2}{10}) + \frac{1}{10} \cdot (-\frac{x^2}{10})^2 + \frac{1}{10} \cdot (-\frac{x^2}{10})^3 + \frac{1}{10} \cdot (-\frac{x^2}{10})^4 \dots \\ &= \frac{1}{10} - \frac{x^2}{100} + \frac{x^4}{1000} - \frac{x^6}{10000} + \frac{x^9}{100000} - + \dots\end{aligned}$$

We can verify this by finding the same Maclauren Series through differentiation:

Given that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n,$$

the quartic Maclaurin polynomial $P_4(x)$ is found as follows:

$f^{(n)}(x)$	$f^{(n)}(0)$
$f^{(0)}(x) = \dfrac{1}{10+x^2}$	$\dfrac{1}{10}$
$f^{(1)}(x) = -\dfrac{2x}{(10+x^2)^2}$	0
$f^{(2)}(x) = \dfrac{6x^2-20}{(10+x^2)^3}$	$-\dfrac{1}{5}$
$f^{(3)}(x) = -\dfrac{24x(x^2-10)}{(10+x^2)^4}$	0
$f^{(4)}(x) = \dfrac{120(x^4-20x^2+20)}{(10+x^2)^5}$	$\dfrac{1}{125}$

$$
\begin{aligned}
P_4(x) &= \frac{\frac{1}{10}}{0!}x^0 + \frac{0}{1!}x^1 + \frac{-\frac{1}{5}}{2!}x^2 + \frac{0}{3!}x^3 + \frac{\frac{1}{125}}{4!}x^4 \\
P_4(x) &= \frac{1}{10} - \frac{1}{100}x^2 + \frac{1}{1000}x^4
\end{aligned}
$$

We can now see that the original sequence constitutes the terms of this Maclaurin series with π in place of x.

2.5.2 Definite Integral approximation through Maclaurin series

If we use integration to get $\int f(x)\mathrm{d}x$, we will find:

$$
\begin{aligned}
\int f(x)\mathrm{d}x &= \int \frac{1}{10+x^2}\mathrm{d}x \\
&= \frac{1}{10}\int \frac{1}{1+\frac{x^2}{10}}\mathrm{d}x \\
&= \frac{1}{10}\int \frac{1}{1+\left(\frac{x}{\sqrt{10}}\right)^2}\mathrm{d}x
\end{aligned}
$$

Let $u = \dfrac{x}{\sqrt{10}}$:

$$
\begin{aligned}
\frac{1}{10}\int \frac{1}{1+\left(\frac{x}{\sqrt{10}}\right)^2}\mathrm{d}x &= \frac{1}{10}\int \frac{1}{1+u^2}\mathrm{d}u \\
\int f(x)\mathrm{d}x &= \frac{1}{\sqrt{10}}\arctan\left(\frac{x}{\sqrt{10}}\right) + C
\end{aligned}
$$

As we have found above, $f(x)$ can also be defined/expressed as

$$f(x) = \frac{1}{10} - \frac{x^2}{100} + \frac{x^4}{1000} - + \dots$$

Finding the indefinite integral of $f(x)$ can then be done by antiderivating it's infinite Maclaurin polynomial as follows:

$$\begin{aligned}
\int f(x)\mathrm{d}x &= \int \left(\frac{1}{10} - \frac{x^2}{100} + \frac{x^4}{1000} - + \dots \right) \mathrm{d}x \\
&= \frac{1 \cdot x^{0+1}}{10 \cdot 1} - \frac{x^{2+1}}{100 \cdot 3} + \frac{x^{4+1}}{1000 \cdot 5} - + \dots \\
&= \frac{x}{10} - \frac{x^3}{300} + \frac{x^5}{5000} - + \dots \\
&= \sum_{n=0}^{\infty} \left((-1)^{n-1} \cdot \frac{x^{2n+1}}{(2n+1) \cdot 10^n} \right) + C
\end{aligned}$$

Conversely, if we use integration to find $\int_0^{\pi} f(x)\mathrm{d}x$, we get:

$$\begin{aligned}
\int_0^{\pi} f(x)\mathrm{d}x &= \int_0^{\pi} \frac{1}{10 + x^2} \mathrm{d}x \\
&= \left[\frac{1}{\sqrt{10}} \arctan\left(\frac{x}{\sqrt{10}} \right) \right]_0^{\pi} \\
&= \frac{1}{\sqrt{10}} \arctan\left(\frac{\pi}{\sqrt{10}} \right) - \frac{1}{\sqrt{10}} \arctan\left(\frac{0}{\sqrt{10}} \right) \\
&= \frac{1}{\sqrt{10}} \arctan\left(\frac{\pi}{\sqrt{10}} \right) - 0 \\
&= \frac{1}{\sqrt{10}} \arctan\left(\frac{\pi}{\sqrt{10}} \right) \approx 0.2473\dots
\end{aligned}$$

Now that we know $\int f(x)\mathrm{d}x = \sum_{n=0}^{\infty} \left((-1)^{n-1} \cdot \frac{x^{2n+1}}{(2n+1) \cdot 10^n} \right) + C$, we can use it to approximate $\int_0^{\pi} f(x)\mathrm{d}x$ as well:

$$\begin{aligned}
\int_0^{\pi} f(x)\mathrm{d}x \sum_{n=0}^{\infty} &= (-1)^{n-1} \cdot \frac{(\pi)^{2n+1}}{(2n+1) \cdot 10^n} - \sum_{n=0}^{\infty} = (-1)^{n-1} \cdot \frac{(0)^{2n+1}}{(2n+1) \cdot 10^n} \\
&= \left(\frac{(\pi)}{10} - \frac{(\pi)^3}{300} + \frac{(\pi)^5}{5000} - \frac{(\pi)^7}{70000} + - \dots \right) - (0) \\
&\approx \frac{\pi}{10} - \frac{\pi^3}{300} + \frac{\pi^5}{5000} - \frac{(\pi)^7}{70000} \approx 0.228862
\end{aligned}$$

Compared to the result definite integral, the approximation is a little bit lower: 0.2473 > 0.228862. It's not very far but it's not close enough to be a good, reliable approximation. Adding more terms to the polynomial will allow the approximation to get ever closer to the exact value.

If we now try to get $\displaystyle\int_0^\infty f(x)\mathrm{d}x$ through integration, we get:

$$\begin{aligned}
\int_0^\infty f(x)\mathrm{d}x &= \int_0^\infty \frac{1}{10+x^2}\mathrm{d}x \\
&= \left[\frac{1}{\sqrt{10}}\arctan\left(\frac{x}{\sqrt{10}}\right)\right]_0^\infty \\
&= \frac{1}{\sqrt{10}}\arctan\left(\frac{\infty}{\sqrt{10}}\right) - \frac{1}{\sqrt{10}}\arctan\left(\frac{0}{\sqrt{10}}\right) \\
&= \frac{1}{\sqrt{10}}\arctan\left(\frac{\infty}{\sqrt{10}}\right) - 0 \\
&= \lim_{x\to\infty}\left(\frac{1}{\sqrt{10}}\arctan\left(\frac{x}{\sqrt{10}}\right)\right) \\
&= \frac{1}{\sqrt{10}}\cdot\frac{\pi}{2} \\
&= \frac{\pi}{2\sqrt{10}}
\end{aligned}$$

Even though we could alternatively use the Maclaurin series in the previous cases, new series we found to approximate for $\displaystyle\int f(x)\mathrm{d}x$ cannot be used to find $\displaystyle\int_0^\infty f(x)\mathrm{d}x$. The approximation will never get the exact output after plugging values such as infinity. Furthermore, infinity is, by definition, infinitely far from 0. This means that it is not comprised in the radius of convergence of the series.

3 Conclusion

Calculus 2 was a honestly challenging topic that kept me engaged and at bay all along. I really enjoyed getting to understand another perspective on how to find volumes of objects from simple cubes to certain more objects resulting from combination of different shapes.

Apart from that, I really enjoyed the process of learning how to write in LaTeXand how to join complex mathematical symbols, expressions, and formulae into a clear and understandable paper.

Through all the rigorous work we've done this semester, I have learned that if something is possibly understandable, I will surely get it too if I take the time to understand why I don't understand the topic. This class was a real test of self-introspection as well as self-discipline.

As an aspiring computer science major, I have always seen mathematics as a topic within the essence of my domain. The concepts and intuitions I've learned in this class are what I will surely never lose. I will now be able to apply them to help me solve various problems in computer science.

Academically, I took this course because it opens me to other alleys of mathematics that are only waiting for me to grasp and tackle. I am contemplating to have math as a minor but I am not sure of that yet. What I am sure of is that this course has personally encouraged me to keep my door always open for maths.

As a final note, I would just like to thank you for being incredibly patient throughout this whole semester. It has been one full of frustration and obstacles, in terms of physical and emotional health and I would like to share my appreciation for making things bearable for me and the whole class. I hope to eventually take one of your classes again, in a hopefully more normal setting with less health-related concerns.

Any final thoughts?